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Dynamics of Extreme Black Holes and Massive String States

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In a recent paper, Duff and Rahmfeld argued that certain massive $N_R = 1/2$ states of the four-dimensional heterotic string correspond to extreme black hole solutions. We provide further, dynamical, evidence for this identification by comparing the scattering of these elementary string states with that of the corresponding extreme black holes, in the limit of low velocities.

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1. Introduction

A particle whose mass is greater than the Planck mass possesses a Schwarzschild radius that is greater than its Compton wavelength. As a result even within quantum mechanics, such a particle may exhibit an event horizon and thus behave like a black hole [1]. It seems, however, that testing this conjecture requires an understanding of quantum gravity. Since string theory is a candidate theory of quantum gravity and it predicts the existence of states with Planck scale masses, it should provide a consistent framework in which the above conjecture might be tested in a concrete manner. Recently the suggestion was made that certain massive excitations of four-dimensional superstrings are indeed black holes [2]. In subsequent work [3], this claim was supported by the discovery that certain extreme black holes could be identified with electrically charged states in the Schwarz-Sen spectrum, which in turn can be identified with elementary string states [4]. On the basis of S-duality, the corresponding solitonic magnetically charged spectrum conjectured by Schwarz and Sen [4] would also be described by extreme black holes [3].

In this paper, we provide dynamical evidence in support of this conjecture by comparing the tree-level scattering amplitudes of the massive string states of the Schwarz-Sen spectrum discussed in [3] with the computation of the low velocity scattering of the corresponding extreme black holes discussed in [2]. In section 2, we summarize the main features of the above conjecture as presented in [3]. In section 3, we summarize the results of Shiraishi [5] using Manton's prescription [6] for the computation of the metric on moduli space for the extreme black holes. In section 4, we compute the scattering amplitudes of the corresponding massive string states. We conclude with a discussion in section 5.

2. Massive String States as Extreme Black Holes

Consider the four-dimensional heterotic string theory arising from toroidal compactification. For a generic point in the moduli space, the low energy effective theory is $N = 4$ supergravity coupled to twenty-two abelian vector multiplets. Thus the bosonic fields include the metric, the dilaton, the axion, 28 abelian gauge fields, and 132 scalar moduli fields. This theory then contains a rich array of black hole solutions with electric and/or magnetic charges [7]. Of the static charged solutions, a special subset have been identified as involving a single $U(1)$ vector field and a single scalar, as well as the metric [2]. These solutions can then be regarded as solutions for a theory described by

$$I = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-a\phi}F^2\right] \ . \tag{2.1}$$

Here, F is a linear combination of the field strengths and their duals appearing in the original effective action, while ϕ represents an appropriate linear combination of the dilaton and the moduli fields. Within field theory, static black hole solutions have been found for arbitrary a by Gibbons [8]. In the context of low energy heterotic string theory, solutions have been identified which correspond to fixing the above scalar-Maxwell coupling to $a = 0$, $1/\sqrt{3}$, 1, or $\sqrt{3}$. (We will assume $a \geq 0$ without loss of generality in the following.) The case $a = 0$ yields the Reissner-Nordstrom solution, for which one actually has $\phi = constant$, and was recognized to be a solution of string theory in [2]. The dilaton black hole [9] arises for $a = 1$. The $a = \sqrt{3}$ case corresponds to the Kaluza-Klein black hole [2] and the "H-monopole" black hole [2,10], which are related to each other by T-duality. The $a = 1/\sqrt{3}$ black hole has also been more recently recognized to be a solution of string theory [11].

In the following, we will be interested in electrically charged extreme black holes, since only extreme black holes possess zero Hawking temperature and thus may be candidates for identification with elementary particles. For spherically symmetric black holes (*i.e.,* with vanishing angular momentum), the extreme solutions for the action (2.1) satisfy

$$(Gm)^2 = \frac{Q^2}{4(1+a^2)}, \tag{2.2}$$

where m is the ADM mass, and Q, the electric charge where $Q \equiv \oint_\infty e^{-a\phi}\tilde{F}/4\pi$ (where the "tilde" denotes the Hodge dual). For such extreme black holes with like charges, the repulsive electrostatic force is precisely balanced by the attractive static forces of the gravitational and scalar fields. Hence it is possible to construct static multi-extreme black hole solutions for the action (2.1). These have long been known for the Reissner-Nordstrom case (*i.e.,* $a = 0$) [12], and also for the dilaton black holes (*i.e.,* $a = 1$) [9] and the "H-monopoles" [13]. Shiraishi [5] found multi-extreme black hole solutions for arbitrary a, but recall only the values of $a = 0, (1/\sqrt{3}), 1, \sqrt{3}$ have so far been demonstrated to arise from heterotic string compactifications. The precise cancellation of forces no longer occurs when the extreme black holes are in motion, and so they will scatter in a nontrivial way. The latter will be the focus of the discussion in section 3.

Within the full four-dimensional string theory, Schwarz and Sen produced a spectrum of states satisfying a certain Bogomol'nyi bound, which is invariant under both $O(6, 22; Z)$ and $SL(2, Z)$ [4]. Of these, a subset can correspond to elementary string states. The latter carry only electric charge in which case the Bogomol'nyi bound can be reduced to [4,3]

$$(Gm)^2 = \frac{1}{16}Q^a(I+L)_{ab}Q^b = \frac{1}{8}(Q_R)^2 \tag{2.3}$$

where the right- and left-projections of the charge vector are defined as $Q_R = \frac{1}{2}(I+L)Q$ and $Q_L = \frac{1}{2}(I-L)Q$ with L the invariant metric on $O(6,22)$, and Q^a the charges of the 28 abelian gauge fields. For elementary string states, $Q_{L,R}$ are related to the internal momenta of the left- and right-movers.

The mass of a heterotic string state in the Neveu-Schwarz sector (which is degenerate with the Ramond sector) is given by[1]

$$m^2 = (\alpha_R)^2 + 2N_R - 1 = (\alpha_L)^2 + 2N_L - 2 \ , \tag{2.4}$$

where $N_{L,R}$ are the left- and right-oscillator numbers, and $\alpha_{L,R}$ are the internal momenta for the left- and right-moving modes. Given $Q^2_{L,R} = 8G^2\,\alpha^2_{L,R}$, a comparison of (2.3) and (2.4) shows that the string states satisfying the Bogomol'nyi bound have $N_R = 1/2$. Given the correspondence of the charge vector with the internal momenta, as well as the masses, these elementary massive $N_R = 1/2$ string states and their superpartners have the correct quantum number to fit into the Schwarz-Sen spectrum [4].

In more recent work, Duff and Rahmfeld [3] show that a subset of these $N_R = 1/2$ states in the Schwarz-Sen spectrum may also be identified as the extreme limits of certain black hole solutions. For these solutions, the low-energy string action can be truncated to (2.1) and the scalar-Maxwell parameter is given by $a = \sqrt{3}$ for $N_L = 1$ and $a = 1$ for $N_L > 1$ and $\alpha_L = 0$. In particular, the charge vector $Q^a = 2\sqrt{2}G\,\delta^{a,1}$ was shown to correspond to an $a = \sqrt{3}$ black hole. This choice satisfies $Q^2_L = Q^2_R$ and hence $N_L = 1$ in (2.4). From (2.3) it follows that the mass is given by $m^2 = 1/2 = Q^2/16G^2$, which coincides with (2.2) in the extreme limit [3]. $O(6,22)$ transformations and rescaling the charge vector then yield all other charge vectors satisfying $Q^2_L = Q^2_R$, and hence all $N_R = 1/2$, $N_L = 1$ states should correspond to $a = \sqrt{3}$ black holes. Similarly, a particular $a = 1$ black hole was shown to correspond to $Q^a = 2\sqrt{2}G(\delta^{a,1} + \delta^{a,7})$. In this case, one has $m^2 = 2 = Q^2/8G^2$ which coincides with (2.2) in the extreme limit for $Q^2 = 16G^2$. In this case, $Q_L = 0$. Again, $O(6,22)$ transformations and rescaling the charge vector will yield all charge vectors satisfying these two conditions, and hence all $N_R = 1/2$, $N_L > 1$ and $\alpha_L = 0$ states correspond to $a = 1$ black holes. Other states with $N_L > 1$ and $\alpha_L \neq 0$ should also be extreme black holes, but a truncation to an effective action of the form (2.1) is not possible. Note that neither $a = 0$ [3] nor $a = 1/\sqrt{3}$ extreme black holes belong to the spectrum. Furthermore, the $a = 1$ extreme dilaton black holes of Ref. [9] also do not belong to the spectrum [3].

[1] We adopt the normalization $\alpha' = 2$ throughout the paper.

3. Extreme Black Holes in Slow Motion

The existence of static multi-soliton solutions, of which the multi-extreme black hole solutions described in the previous section are examples, relies on the cancellation of the scalar, vector and tensor exchange forces (the so-called "zero-force" condition). If the solitons are given velocities, however, the zero-force condition ceases to hold and dynamical, velocity-dependent forces arise. The full time-dependent equations of motion that result are highly nonlinear and in general very difficult to solve. In the absence of exact time-dependent multi-soliton solutions, Manton's method [6] for the computation of the metric on moduli space represents a good low-velocity approximation to the exact dynamics of the solitons. Manton's prescription for the study of soliton scattering may be summarized as follows: One begins with a static multi-soliton solution, and gives the moduli characterizing this configuration a time-dependence. One then finds $O(v)$ corrections to the fields by solving the constraint equations of the system with time-dependent moduli. The resultant time-dependent field configuration only satisfies the full time-dependent field equations to lowest order in the velocities, but provides an initial data point for the fields and their time derivatives. Another way of saying this is that the initial motion is tangent to the set of exact static solutions. An effective action describing the motion of the solitons is determined by replacing the solution to the constraints into the field theory action. The kinetic action so obtained defines a metric on the moduli space of static solutions, and the geodesic motion on this metric determines the dynamics of the solitons. This approach was first applied to study the scattering of BPS monopoles [6], and a complete calculation of the corresponding moduli space metric and a description of its geodesics was worked out by Atiyah and Hitchin [14].

Manton's method was subsequently adapted to general relativistic actions by Ferrell and Eardley [15] for the low-energy scattering of extreme Reissner-Nordstrom black holes. More recently, Shiraishi [16] adapted the method of Ref. [15] to obtain the metric on moduli space for generalized multi-black hole solutions of the action (2.1). The resulting effective Lagrangian describing the interactions of N extremally charged black holes is

$$\begin{aligned} L = &-\sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \\ &+ \frac{3-a^2}{16\pi} \int d^3x F(x)^{2(1-a^2)/(1+a^2)} \sum_{i\neq j}^{N} G m_i m_j |\vec{v}_i - \vec{v}_j|^2 \frac{\vec{r}_i \cdot \vec{r}_j}{r_i^3\, r_j^3} \end{aligned} \tag{3.1}$$

where $\vec{r}_i = \vec{x} - \vec{x}_i$, and $\vec{v}_i$ and $\vec{x}_i$ are, respectively, the velocity and position of the i'th black hole. Also,

$$F(x) = 1 + (1 + a^2) \sum_{i=1}^{N} \frac{Gm_i}{r_i} \ . \tag{3.2}$$

As usual in the Manton method, this effective action represents the leading terms up to $O(v^2)$ in a small velocity expansion, and so neglects the effects of any radiation fields which would only contribute at higher orders. The first two terms in (3.1) correspond to the expected free particle Lagrangian to $O(v^2)$. The remaining term is the interaction Lagrangian which as expected vanishes as the relative velocities go to zero. In general, this contribution is highly nonlinear involving up to N-body interactions. Collecting all of the $O(v^2)$ terms yields a metric on the moduli space of these N black hole configurations.

The above interactions simplify for two values of the scalar-Maxwell coupling, $a = \sqrt{3}$ and $a = 1$, which are precisely the values of interest in the present paper. The effective Lagrangian for extreme $a = \sqrt{3}$ black holes reduces to the free terms only [5,16]

$$L = -\sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \ . \tag{3.3}$$

In other words, the leading order velocity-dependent (*i.e.*, $O(v^2)$) dynamical force between the black holes is zero, and the low-velocity scattering is trivial. Thus one infers that the metric on the moduli space of these $a = \sqrt{3}$ extreme black holes is flat. A similar flat metric has been found for H-monopoles [17], fundamental strings [18] and $D = 10$ fivebranes [19]. In fact, one can show that a flat metric describes the motion of all κ-symmetric p-branes [20].

For $a = 1$, the Lagrangian (3.1) simplifies in that it only involves two-body interactions. Thus the effective Lagrangian is easily determined to be simply [16]

$$L = -\sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 + \frac{1}{2} \sum_{i \neq j}^{N} G m_i m_j \frac{|\vec{v}_i - \vec{v}_j|^2}{|\vec{x}_i - \vec{x}_j|} \ . \tag{3.4}$$

In this case, the nontrivial interaction leads to a center-of-mass deflection angle for the scattering of two $a = 1$ black holes: $\theta = 2 \tan^{-1}(GM/b)$, where $M = m_1 + m_2$ is the total mass and b is the impact parameter [16]. The resulting differential cross section then has the Rutherford scattering form [16]

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \frac{(GM)^2}{\sin^4(\theta/2)}. \tag{3.5}$$

Note that while Shiraishi's work describes the dynamics of black hole solutions for the truncated action (2.1), we are interested in black hole solutions of the low energy effective string action. In this case, we must consider the possibility that the time-dependent solutions will involve contributions from the other bosonic fields in the full theory. It is not hard to show that the motion of the black holes does not induce nonvanishing contributions from the other moduli scalars or $U(1)$ vectors. Similarly for $a=\sqrt{3}$, the axion remains vanishing. On the other hand for $a=1$, the motion of the black holes will lead to a nontrivial axion. The effective action which must be considered then is

$$\begin{aligned} I = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,\Big[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-\phi}F^2 \\ -\frac{1}{2}e^{2\phi}(\partial\rho)^2 + \frac{1}{8}\rho\,\varepsilon^{abcd}F_{ab}F_{cd}\Big] \end{aligned} \tag{3.6}$$

where ρ is the scalar axion field. The equation of motion for the latter is

$$\nabla^a(e^{2\phi}\nabla_a\rho) = -\frac{1}{8}\varepsilon^{abcd}F_{ab}F_{cd} = \vec{E}\cdot\vec{B} \tag{3.7}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, respectively, as the spatial three vectors. Now for static electrically charged black holes, the absence of any magnetic fields means that the axion is consistently set to $\rho=0$. When these charged black holes are in relative motion though, the presence of both electric and magnetic fields requires that a nonvanishing axion field for a consistent solution.

However, we will now argue that effective Lagrangian (3.4) remains valid as a leading approximation to the full results for stringy black holes interacting at large separations. For small relative velocities, the magnetic field will be $O(v)$ and so by (3.7), the axion field is of the same order. The key observation is that since the magnetic fields are proportional to the electric charges, and hence the black hole masses by (2.2), the induced axion field is proportional to products of masses, $G^2m_im_j$. Thus when the axion field is substituted back into the action (3.6), following the Manton method, the resulting interactions schematically take the form $G^3m^4v^2/r^3$ (at least to leading order). Here, the $1/r^3$ dependence on the black hole separations is inferred by dimensional analysis. Similarly the modifications of the metric, dilaton and gauge fields induced by the axion can only lead to modifications of the same or higher order in Gm/r. Thus if we consider making an expansion of the interaction lagrangian in Gm/r for black holes interaction at large separations, we see that Shiraishi's results (3.4) remain the leading contribution, and that the axion will only modify the higher order interactions. Our calculations of string scattering will only be sensitive to the leading $O(1/r)$ interaction, and hence the above results are sufficient for our purposes.

4. Amplitudes for Massive String States

We now consider the scattering of massive heterotic string states characterized by: (i) $N_R = \frac{1}{2}$, $N_L = 1$ and hence $\alpha_L^2 = \alpha_R^2$; and (ii) $N_R = \frac{1}{2}$, $N_L > 1$ and $\alpha_L = 0$. These, of course, are expected to correspond to the $a = \sqrt{3}$ and $a = 1$ extreme black holes as discussed above. We will calculate the corresponding four-point closed string amplitudes using the methods of Ref. [21], where it was shown that the left- and right-moving world-sheet modes could be treated completely independently. The final closed string amplitudes are constructed by sewing together two open string amplitudes for these independent modes. The four particle interactions described by the full amplitudes include contributions from all possible massive string states, as well as the exchange of the massless fields appearing in the low-energy effective action (*i.e.,* gravitons, vectors and scalars). The results of section 3 for the scattering of extreme black holes only account for the latter massless particle exchange. The former results also only describe the scattering of black holes corresponding to a fixed embedding of the fields in (2.1) into the full low energy string theory. Of course, the previous analysis was also limited to low velocity scattering. Hence in the string amplitude, we choose particles 1 and 4 to be identical string states with the same internal momentum and polarizations, and we make a similar choice for particles 2 and 3. Then we arrange the spacetime kinematics of these particles to yield low velocity elastic scattering. Finally the string amplitude will be examined for contributions with a t-channel pole, which will correspond to the exchange of massless particles.

Following the procedure of [21] we split our vertex operators into left- and right-movers. Since our string states are all in the Neveu-Schwarz sector of the right movers with $N_R = 1/2$, the right-moving excitations can be considered as a massless superstrings, in a ten-dimensional sense. Similarly the left-states correspond to (twenty-six-dimensional) bosonic strings with masses $M_L^2 = 2(N_L - 1)$. For each state then, there is a left- and right-momentum vector with 26 and 10 components, respectively. In accord with our choice of kinematics, $p^\mu_{1L,R}$ and $p^\mu_{2L,R}$ are the incoming momenta and $-p^\mu_{3L,R}$ and $-p^\mu_{4L,R}$ are the outgoing momenta. Hence $p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0$ for both left- and right-momenta. The spacetime four-momenta, which we denote k_i^μ, correspond to the first four components of either $p^\mu_{iL,R}$. Hence we have:

$$
\begin{aligned}
p^\mu_{1L,R} &= (\ \ E, \ \ E\vec{v}_1, \ \ \alpha_{1L,R}),\\
p^\mu_{2L,R} &= (\ \ \tilde{E}, -\tilde{E}\vec{v}_2, \ \ \alpha_{2L,R}),\\
p^\mu_{3L,R} &= (-\tilde{E}, \ \ \tilde{E}\vec{v}_3, -\alpha_{2L,R}),\\
p^\mu_{4L,R} &= (-E, -E\vec{v}_4, -\alpha_{1L,R}),
\end{aligned}
\tag{4.1}
$$

where $\vec{v}_i$ are the three-dimensional spatial velocities, and $\alpha_{iL,R}$ are the internal left- and right-momenta, respectively. We have used energy-momentum conservation to write $E_1 = E_4 = E$ and $E_2 = E_3 = \tilde{E}$, and further we have $v_1^2 = v_4^2 = v^2$ and $v_2^2 = v_3^2 = w^2$. For comparison to the low velocity scattering in the previous section, we restrict $v^2, w^2 << 1$. Working in the center-of-mass frame in the four-dimensional spacetime, $E\vec{v}_1 = \tilde{E}\vec{v}_2$ and $E\vec{v}_4 = \tilde{E}\vec{v}_3$. Note that $\vec{v}_1 \cdot \vec{v}_4 = v^2 \cos\theta$ and $\vec{v}_2 \cdot \vec{v}_3 = w^2 \cos\theta$, where θ is the scattering angle in the center-of-mass frame. Further, one has

$$E_i^2(1 - v_i^2) = m_i^2 = \alpha_{iR}^2 = \alpha_{iL}^2 + 2(N_{iL} - 1) \ . \tag{4.2}$$

Note that for slow motion scattering with $v_i^2 << 1$, one has $E_i = m_i + \frac{1}{2}m_i v_i^2 + O(v_i^4)$.

We choose all the internal momenta to be parallel (*i.e.*, $\alpha_{2R} = \beta\alpha_{1R}$ and $\alpha_{2L} = \beta\alpha_{1L}$). This ensures that our string states all couple to the scalars and gauge fields in the same way. Note that this imposes $m_2 = \beta m_1$ and $(N_{2L} - 1) = \beta^2(N_{1L} - 1)$. Similarly to compare to elastic scattering of identical black holes, we must choose the polarization tensors to be the same for states 1 and 4, as well as for 2 and 3 (*i.e.*, $\zeta^1 = \zeta^4$ and $\zeta^2 = \zeta^3$.) These polarization tensors are schematically represented as tensor products of separate polarization tensors for the left- and right-movers (*i.e.*, $\zeta^i = \zeta^{iR} \otimes \zeta^{iL}$), which satisfy $p_{iR} \cdot \zeta_{iR} = 0 = p_{iL} \cdot \zeta_{iL}$. For comparison with spherically symmetric black holes, the string states should be scalars in the four-dimensional spacetime. The simplest choice which satisfies this restriction, and which we make below, is to choose the polarization tensors only to take nonvanishing values for the internal directions.

It is useful to write out the higher-dimensional Mandelstam variables for the above configuration. For the right-movers,

$$\begin{aligned}
s_R &= -(p_{1R} + p_{2R})^2 = (E + \tilde{E})^2 - (1+\beta)^2\alpha_{1R}^2 = O(v^2), \\
t_R &= -(p_{2R} + p_{3R})^2 = -2E^2v^2(1 - \cos\theta), \\
u_R &= -(p_{1R} + p_{3R})^2 = (E - \tilde{E})^2 - 2E^2v^2(1 + \cos\theta) - (1-\beta)^2\alpha_{1R}^2 = O(v^2)
\end{aligned} \tag{4.3}$$

with $s_R + t_R + u_R = 0$. Similarly, for the left-movers,

$$\begin{aligned}
s_L = -(p_{1L}^\mu + p_{2L}^\mu)^2 =& (E + \tilde{E})^2 - (1+\beta)^2\alpha_{1L}^2 \\
=& 2(1+\beta)^2(N_{1L} - 1) + O(v^2), \\
t_L = -(p_{2L}^\mu + p_{3L}^\mu)^2 =& -2E^2v^2(1 - \cos\theta), \\
u_L = -(p_{1L}^\mu + p_{3L}^\mu)^2 =& (E - \tilde{E})^2 - 2E^2v^2(1 + \cos\theta) - (1-\beta)^2\alpha_{1L}^2 \\
=& 2(1-\beta)^2(N_{1L} - 1) + O(v^2),
\end{aligned} \tag{4.4}$$

with $s_L+t_L+u_L=4(1+\beta^2)(N_{1L}-1)$. Note that the spacetime Mandelstam variable t coincides with that of either the left- or right-momenta (*i.e.*, $t=-(k_2+k_3)^2=t_R=t_L$). On the other hand in general, the spacetime Mandelstam variables s and u need not coincide with those of the left- and right-momentum vectors. For later purposes, we note that

$$\begin{aligned} s_R &= -(k_1+k_2)^2-(1+\beta)^2\alpha_R^2 = s-(1+\beta)^2 m_1^2 \\ s_L &= -(k_1+k_2)^2-(1+\beta)^2\alpha_L^2 = s_R+2(1+\beta)^2(N_{1L}-1) \\ u_R &= -(k_1+k_3)^2-(1-\beta)^2\alpha_R^2 = u-(1-\beta)^2 m_1^2 \\ u_L &= -(k_1+k_3)^2-(1-\beta)^2\alpha_L^2 = u_R+2(1-\beta)^2(N_{1L}-1). \end{aligned} \tag{4.5}$$

Following the methods of Ref. [21] (see also [22]), the four-point amplitude for heterotic string states can be expressed in terms of four-point amplitudes for the open left-moving bosonic and open right-moving supersymmetric states:

$$A_{4,het}\approx \sin(\pi t_R/2)\,A_{4,sup}(t_R,u_R)\,A_{4,bos}(s_L,t_L), \tag{4.6}$$

where $A_{4,sup}(t_R,u_R)$ corresponds to a t-u channel open superstring amplitude, and $A_{4,bos}(s_L,t_L)$ corresponds to an s-t channel open bosonic string amplitude. Note that the term $\sin(\pi t_R/2)$, which is needed to "sew" the two open string amplitudes, could equally well have been written as $\sin(\pi t_L/2)$, since $t_L=t_R$ for the present kinematics. In fact this would be possible in general since it is always the case that the difference between corresponding right- and left-momentum Mandelstam variables is always an integer multiple of 8. Hence the sewing factor is always identical no matter from which side the momenta are chosen. In the small velocity limit, the sewing term is of $O(v^2)$ for all the cases we consider, and we may write $\sin(\pi t_R/2)\approx \pi t_R/2$.

For the right-moving superstring contribution, one has

$$A_{4,sup}(t_R,u_R)=\frac{\Gamma(-t_R/2)\Gamma(-u_R/2)}{\Gamma(1+s_R/2)}K_R(p_{iR},\zeta^{jR}) \tag{4.7}$$

where K_R is the kinematic factor for four massless superstring vector states. Recall that for simplicity, we will choose the polarization vectors to point in the internal directions. Further note that since the internal momentum vectors are all parallel, one has $p_{iR}\cdot\zeta^{jR}=0$ for any i and j. This greatly simplifies the analysis since in this case the kinematic factor reduces to

$$K_R=-\frac{1}{4}(s_R\,t_R\,\zeta^{1R}\cdot\zeta^{3R}\,\zeta^{2R}\cdot\zeta^{4R}+s_R\,u_R\,\zeta^{2R}\cdot\zeta^{3R}\,\zeta^{1R}\cdot\zeta^{4R}+t_R\,u_R\,\zeta^{1R}\cdot\zeta^{2R}\,\zeta^{3R}\cdot\zeta^{4R})\ . \tag{4.8}$$

Note that this factor is $O(v^4)$ since s_R, t_R and u_R are each $O(v^2)$ — see eq. (4.3). The Gamma functions in (4.7) give poles in t_R and u_R, and so the net result is that $A_{4,sup}(t_R, u_R)$ is of order 1.

Combining the superstring and sewing factors, one has

$$\begin{aligned}\sin(\pi t_R/2)\, A_{4,sup}(t_R, u_R) \approx -\frac{\pi}{2}\bigg(&\frac{s_R\, t_R}{u_R}\, \zeta^{1R}\!\cdot\zeta^{3R}\, \zeta^{2R}\!\cdot\zeta^{4R} + s_R\, \zeta^{2R}\!\cdot\zeta^{3R}\, \zeta^{1R}\!\cdot\zeta^{4R} \\ &+ t_R\, \zeta^{1R}\!\cdot\zeta^{2R}\, \zeta^{3R}\!\cdot\zeta^{4R}\bigg)\ .\end{aligned} \tag{4.9}$$

This final result is $O(v^2)$, thus if the total amplitude is to be $O(1)$, $A_{4,bos}$ must supply an $O(v^{-2})$ pole. Further if we are to identify a t-channel pole, it must arise as a $1/t_L = 1/t_R$ factor in $A_{4,bos}$. The analysis of the bosonic string factor differs for the two cases under consideration and so we separate the discussion:

(i) $N_R = \frac{1}{2}$ and $N_L = 1$

With $N_L = 1$ the left-movers also correspond to four massless vectors (in a twenty-six-dimensional sense). In this case, a number of simplifications occur for the kinematic variables. In particular since $N_{1L} = N_{2L} = 1$, we see from eq. (4.4) that $s_L = O(v^2) = u_L$ (and $s_L + t_L + u_L = 0$). The amplitude then reduces to

$$A_{4,bos}(s_L, t_L) = \frac{\Gamma(-s_L/2)\Gamma(-t_L/2)}{\Gamma(1 + u_L/2)}\, \tilde{K}_L(p_{iR}, \zeta_{jR}) \tag{4.10}$$

where $\tilde{K}_L$ is the appropriate bosonic string kinematic factor. Again all of the left internal momentum vectors are parallel, and so with internally pointing polarization vectors, $(p_{iR}\cdot\zeta^{jL}) = 0$ for any i and j. With these conditions the kinematic factor takes the same form as for the superstring with

$$\tilde{K}_L = -\frac{1}{4}(s_L\, t_L\, \zeta^{1L}\!\cdot\zeta^{3L}\, \zeta^{2L}\!\cdot\zeta^{4L} + s_L\, u_L\, \zeta^{2L}\!\cdot\zeta^{3L}\, \zeta^{1L}\!\cdot\zeta^{4L} + t_L\, u_L\, \zeta^{1L}\!\cdot\zeta^{2L}\, \zeta^{3L}\!\cdot\zeta^{4L})\ . \tag{4.11}$$

Note that this factor is again $O(v^4)$ while the Gamma functions in (4.10) contribute poles in s_L and t_L, each of $O(v^{-2})$. Thus just as for the right-movers, the net result is $O(1)$.

This leaves the total heterotic string amplitude as $O(v^2)$, which indicates that there is no scattering of these string states, to leading order in the low velocity approximation. Since a flat metric was found for the moduli space of $a = \sqrt{3}$ extreme black holes, the result supports the identification of these string states and $a = \sqrt{3}$ black holes.

(ii) $N_R = \frac{1}{2}$, $N_L > 1$ and $\alpha_L = 0$

With $N_L > 1$, the left-movers also correspond to four massive tensor states (in the twenty-six-dimensional sense). In the following, we will assume that $N_{2L} \geq N_{1L}$ and hence $\beta \geq 1$. Explicit results for the required amplitudes are not readily available in the literature, however it is not difficult to perform the necessary calculations. We represent the left-moving states with the vertex operators

$$V_i = \zeta^{iL}_{\alpha\cdots\beta}\, \partial X^{\alpha} \cdots \partial X^{\beta} e^{ip_{iL}\cdot X} \tag{4.12}$$

where $\zeta^{1L} = \zeta^{4L}$ and $\zeta^{2L} = \zeta^{3L}$ are symmetric, traceless tensors with N_{1L} and N_{2L} indices, respectively. Again both polarization tensors only take nonvanishing values for internal directions and since $\alpha_{iL} = 0$, the contraction of any of the momenta p_{iL} with any indices on the polarization tensors vanishes. This simplifies the corresponding amplitude, and schematically one finds

$$\begin{aligned} A_{4,bos} \approx \int \prod_{i=1}^{4} dx_i \frac{|x_a - x_b||x_a - x_c||x_b - x_c|}{dx_a\, dx_b\, dx_c} \prod_{i>j} |x_i - x_j|^{p_{iL}\cdot p_{jL}} \left[\frac{1}{(x_2 - x_3)^2}\right]^{N_{2L} - N_{1L}} \\ \left[\frac{1}{(x_1 - x_2)^2 (x_3 - x_4)^2} + \frac{1}{(x_1 - x_3)^2 (x_2 - x_4)^2} + \frac{1}{(x_1 - x_4)^2 (x_2 - x_3)^2}\right]^{N_{1L}} \end{aligned} \tag{4.13}$$

The factors missing in the expression above are various contractions of the polarization tensors. One can implicitly keep track of these contractions through the factors of $(x_i - x_j)^{-2}$ (*e.g.*, $(x_1 - x_2)^{-2}$ corresponds to a contraction of a pair of indices between ζ^{1L} and ζ^{2L}). The factor inserted indicates that three of the integration variables are to be fixed. This removes the $SL(2, R)$ divergence in the full integration, and provides the Faddeev-Popov determinant. For the s-t channel, we choose: $x_1 = 0$, $0 \leq x_2 = x \leq 1$, $x_3 = 1$, $x_4 = C \to \infty$. The above expression then reduces to

$$A_{4,bos} \approx \int_0^1 dx\, (1-x)^{p_{2L}\cdot p_{3L}}\, x^{p_{1L}\cdot p_{2L}} \left[\frac{1}{x^2} + 1 + \frac{1}{(1-x)^2}\right]^{N_{1L}} \left[\frac{1}{(1-x)^2}\right]^{N_{2L} - N_{1L}} \tag{4.14}$$

Further we have $p_{1L}\cdot p_{2L} = N_{1L} + N_{2L} - 2 - (s_L/2)$ and $p_{2L}\cdot p_{3L} = 2N_{2L} - 2 - (t_L/2)$. Thus the above expression (4.14) becomes a sum of terms of the form

$$\begin{aligned} I_{ab} &= \int_0^1 dx (1-x)^{(2N_{1L} - 2 - 2a - t_L/2)} x^{(N_{1L} + N_{2L} - 2 - 2b - s_L/2)} \\ &= \frac{\Gamma(2N_{1L} - 1 - 2a - t_L/2)\Gamma(N_{1L} + N_{2L} - 1 - 2b - s_L/2)}{\Gamma(N_{1L} - N_{2L} + 2 - 2(a+b) + u_L/2)} \end{aligned} \tag{4.15}$$

where a and b are non-negative integers with $a + b \leq N_{1L}$. Considering each of these Gamma function factors in turn:

a) $\Gamma(2N_{1L} - 1 - 2a - t_L/2)$: From (4.4), $t_L = O(v^2)$. Hence the argument of this factor is essentially a positive integer, except for the case $a = N_{1L}$ (and $b = 0$). In the latter case, the argument is almost -1, and this factor in the numerator contributes an $O(v^{-2})$ pole. Otherwise this factor is simply a finite constant.

b) $\Gamma(N_{1L} + N_{2L} - 1 - 2b - s_L/2)$: From (4.4), $s_L/2 = (\beta + 1)^2(N_{1L} - 1) + O(v^2)$. Using $(N_{2L} - 1) = \beta^2(N_{1L} - 1)$, this factor becomes $-2\beta(N_{1L} - 1) + 1 - 2b + O(v^2)$. With $\beta \geq 1$ and $N_{1L} \geq 2$, this argument is negative for all choices of a and b. In fact, one can show that it is a negative integer up to $O(v^2)$, and so this factor contributes an $O(v^{-2})$ pole for all values of a and b.

c) $\Gamma(N_{1L} - N_{2L} + 2 - 2(a + b) + u_L/2)$: From (4.4), $u_L/2 = (\beta - 1)^2(N_{1L} - 1) + O(v^2)$. In this case, the argument then becomes $-2(\beta - 1)(N_{1L} - 1) + 2 - 2(a + b) + O(v^2)$. Generically, one can show that this argument is a non-positive integer up to $O(v^2)$, and so this factor in the denominator contributes a zero of $O(v^2)$. [2]

Overall then the generic I_{ab} are finite since the zero in (c) cancels the pole in (b). Hence the corresponding terms do not contribute to the low velocity scattering. The only relevant term with an $O(v^{-2})$ pole is that with $(a, b) = (N_{1L}, 0)$ which yields

$$\begin{aligned} I_{N_{1L},0} =& \frac{\Gamma(-1 - t_L/2)\Gamma(N_{1L} + N_{2L} - 1 - s_L/2)}{\Gamma(-N_{1L} - N_{2L} + 2 + u_L/2)} \\ \approx& \frac{4\beta(N_{1L} - 1)}{t_R} \frac{u_R}{s_R} \end{aligned} \tag{4.16}$$

using (4.5) and $t_L = t_R$. Using $s_R + t_R + u_R = 0$, one can rewrite this expression as

$$I_{N_{1L},0} \approx -4\beta(N_{1L} - 1) \left(\frac{1}{t_R} + \frac{1}{s_R} \right) \tag{4.17}$$

Note that this contribution included all of the factors of $(1 - x)^2 = (x_3 - x_2)^2$ in (4.15) so for the polarization factor in this term one has only contractions between ζ^{2L} and ζ^{3L}, and

[2] The only exception to this behavior is if $\beta = 1$ (*i.e.*, $N_{1L} = N_{2L}$) and $a = b = 0$, in which case, the argument is positive and this factor no longer contributes a zero. With further analysis, one finds no t-channel pole in this case. However since both pairs of ingoing and outgoing states are now identical, one does find an analogous pole corresponding to massless particle exchange in the u-channel where particles 3 and 4 are interchanged.

also between ζ^{1L} and ζ^{4L} from the implicit factors of $(x_4 - x_1)^2$. These are the expected contractions for elastic scattering of $1 \to 4$ and $2 \to 3$.

Now gathering up the relevant terms, the total amplitude yields

$$\begin{aligned} A_{4,het} \approx & 2\beta(N_{1L} - 1)\, \zeta^{1L}\cdot\zeta^{4L}\, \zeta^{2L}\cdot\zeta^{3L} \left(\frac{1}{t_R} + \frac{1}{s_R}\right) \\ & \left(\frac{s_R\, t_R}{u_R}\, \zeta^{1R}\cdot\zeta^{3R}\, \zeta^{2R}\cdot\zeta^{4R} + s_R\, \zeta^{2R}\cdot\zeta^{3R}\, \zeta^{1R}\cdot\zeta^{4R} + t_R\, \zeta^{1R}\cdot\zeta^{2R}\, \zeta^{3R}\cdot\zeta^{4R}\right) \\ \approx & 2\sqrt{N_{1L} - 1}\sqrt{N_{2L} - 1}\frac{s_R}{t_R} + \cdots \end{aligned} \tag{4.18}$$

where we have isolated the only t-channel pole.

Note that in this expression, the first factor maybe rewritten as $2\sqrt{N_{1L} - 1}\sqrt{N_{2L} - 1} = m_1 m_2$. Now within the low velocity approximation, one finds $s_R \approx m_1 m_2 |\vec{v}_1 - \vec{v}_2|^2$. Following [23] in Fourier transforming the t-channel momentum and dividing by $4(m_1 m_2)$ to account for the relativistic normalization of states, one then arrives at a non-relativistic potential describing this t-channel interaction of the form

$$U \approx m_1 m_2 \frac{|\vec{v}_1 - \vec{v}_2|^2}{|\vec{x}_1 - \vec{x}_2|} \tag{4.19}$$

up to numerical prefactors. This potential then coincides precisely with that describing the leading order long range interaction of the $a = 1$ black holes. Thus we have further, dynamical, evidence for the identification of massive string states with black holes.

5. Discussion

Following Duff and Rahmfeld's identification of certain states of the Sen-Schwarz spectrum with extreme black holes, Sen [24] supported this correspondence by arguing that string theory could correct the Bekenstein-Hawking black hole entropy so as to reproduce the logarithm of the density of states of elementary string states. Peet [25] generalized this correspondence as well as the entropy analysis of Sen to black holes and massive string states in higher dimensions. In this paper we presented dynamical evidence for the identification by comparing two seemingly very different methods of scattering: that of Manton's metric on moduli space approximation to the low-velocity dynamics of classical extreme black hole solutions of the low-energy effective action on the one hand, and the string four-point amplitude for the scattering of the corresponding string states on the

other. The fact that the results of these two methods are in agreement is rather remarkable, and points to possible as yet unrealized connections between classical solutions of string theory and states in its spectrum.

Our results can be extended to the dynamics of extremal black holes and corresponding massive string states in higher dimensions as follows: Consider the action in D dimensions

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left[R - \frac{\gamma}{2} (\partial\phi)^2 - \frac{1}{4} e^{-a\gamma\phi} F^2 \right] , \tag{5.1}$$

where $\gamma = 2/(D-2)$. The same Bogomol'nyi bound (2.2) holds in this case. The κ-symmetric extremal black holes with $a = \sqrt{D-1}$ scatter trivially [16] and correspond to string states with $N_R = 1/2$ and $N_L = 1$ in D dimensions, in analogy with the $a = \sqrt{3}$ case in $D = 4$. The $a = 1$ extremal black holes again correspond to $N_R = 1/2$, $N_L > 1$ states with $\alpha_L = 0$ and the low-velocity scattering following Manton's method again matches the corresponding four-point amplitude.

One would also like to extend this analysis to the $N_R = 1/2$ string states with $N_L > 1$ and $\alpha_L \neq 0$. These should correspond to extreme black holes, but a truncation to an effective action of the form (2.1) (or (5.1) for $D > 4$) is not possible. The generalized black hole solutions are known [7,25], but the Manton metric describing their low-velocity scattering remains to be calculated. The calculation of the corresponding string amplitudes is essentially unchanged, and the final amplitudes still carry a factor of $2\beta(N_{1L}-1) = \beta(\alpha_{1R}^2 - \alpha_{1L}^2)$. Keeping in mind that the internal momenta are all parallel in these calculations, one can re-express this factor as $Q_1^T L Q_2$, where again L is the invariant metric on $O(6,22)$. Thus the conjectured identification of string states and black holes would predict that the low-velocity scattering of these black holes is governed by an interaction of the form

$$U \approx Q_1^T L Q_2 \frac{|\vec{v}_1 - \vec{v}_2|^2}{|\vec{x}_1 - \vec{x}_2|^{D-3}} \tag{5.2}$$

in D dimensions.

One possible gap in the correspondence tested here is that there seem to be many string states corresponding to a single extreme black hole. While the black hole solution in the low energy theory are completely fixed once the charge vector is specified, one still has the freedom to specify the polarization tensor for the string states. One possible way around this problem is, following Sen [24], to claim that quantum corrections to the entropy of the black hole would correspond to the entropy of the string states. An alternate suggestion is that when one goes beyond the massless fields in effective action to include

the massive string states and Kaluza-Klein modes of the full theory, one will find new black hole solutions such that there will be a one-to-one correspondence between the black holes and string states. At present, this suggestion has yet to be investigated. A related problem worth considering is the conversion between different polarizations of elementary string states by interactions with the moduli scalars. This process should correspond to the conversion between different extremal black holes, in analogy with the conversion of monopoles into dyons found in the study of BPS monopoles [14].

In comparing the dynamics of black holes and massive string states, we must be careful to remain within the validity of the low-velocity approximation. For example, the metric on moduli space for two $a = 1$ black holes seems to indicate that such black holes will never coalesce [16]. Examining the string amplitude (4.18) though, one does find that there are s-channel poles. These arise from the two incoming particles merging to form a single string state with mass $m = m_1 + m_2 = (1 + \beta)m_1$ and internal momenta $\alpha_R = \alpha_{1R} + \alpha_{2R} = (1 + \beta)\alpha_{1R}$ and $\alpha_L = 0$. Therefore the intermediate state still satisfies $m^2 = \alpha_R^2$ and hence is still a state with $N_R = 1/2$, $N_L > 1$ and $\alpha_L = 0$. Thus the intermediate state should also correspond to another $a = 1$ black hole. The string theory calculation then seems to indicate that $a = 1$ black holes can in fact merge, in apparent contradiction to the Manton scattering results. However, following the discussion of [15], when the relative separation $r \to 0$, the validity of Manton's method breaks down and so in fact there is no contradiction.

Finally, we note that the agreement between a flat metric for the scattering of elementary string solutions and a vanishing four-point amplitude for string winding states in the infinite winding radius limit was found previously in [26] in order to provide dynamical evidence for the identification of elementary string solutions of the equations of motion with macroscopic fundamental winding states. The present comparison for $a = \sqrt{3}$ extremal black holes involves point-like string states and black holes and so requires a different limit. It is interesting that the same phenomenon of trivial scattering, from both viewpoints, appears in the two different scenarios.

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